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Estimates of the Derivatives by the Shimizu-Ahlfors Characteristic Functions

by

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1. Introduction

The Shimizu-Ahlfors characteristic function of f meromorphic in $D_A=\{|z|<A\}$, $0<A\leqq\infty$, is the nondecreasing function

$$T_1(r,f)=\pi^{-1}\int_0^r t^{-1}\left[\iint_{|z|<t} f^{\#}(z)^2dxdy\right]dt\leqq\infty$$

of r, $0<r\leqq A$, where $f^{\#}=|f'|/(1+|f|^2)$, $z=x+iy$. If f is holomorphic in D_A, we set

$$T_2(r,f)=\pi^{-1}\int_0^r t^{-1}\left[\iint_{|z|<t} |f'(z)|^2dxdy\right]dt$$

for r, $0<r\leqq A$, and further if f is bounded, $|f|<1$, then we set

$$T_3(r,f)=\pi^{-1}\int_0^r t^{-1}\left[\iint_{|z|<t} f^{*}(z)^2dxdy\right]dt$$

for r, $0<r\leqq A$, where $f^*=|f'|/(1-|f|^2)$. These are nondecreasing functions of r.

First, for f meromorphic in $D=D_1$, and for r, $0<r\leqq 1$, we set

$$\alpha_1(f)=\sup_{w\in D}(1-|w|^2)f^{\#}(w)\,,\qquad \beta_1(r,f)=\sup_{w\in D}T_1(r,f_w)\,,$$

where

$$f_w(z)=f\left(\frac{z+w}{1+\bar{w}z}\right),\qquad z\in D\,.$$

THEOREM 1. *For f meromorphic in D, and for r, $0<r<1$, we have*

$$\left(\log\frac{1}{\sqrt{1-r^2}}\right)^{-1}\beta_1(r,f)\leqq\alpha_1(f)^2\leqq r^{-2}[\exp\{2\beta_1(r,f)\}-1]\,. \tag{1.1}$$

We call f normal if $\alpha_1(f)<\infty$ [2]. It then follows immediately from (1.1) that

$$\alpha_1(f) < \infty \,. \Leftrightarrow \beta_1(r, f) < \infty \qquad \textit{for each} \quad r \in (0, 1)\,.$$
$$\Leftrightarrow \beta_1(r, f) < \infty \qquad \textit{for an} \quad r \in (0, 1)\,.$$

If $\beta_1(1, f) < \infty$, that is, if $f \in UBC$ [6], then it follows from the right-hand estimate of (1.1) that f is normal as is observed in [6].

Secondly, for f meromorphic in $\boldsymbol{C} = D_\infty$, and for r, $0 < r \leqq \infty$, we set

$$\alpha_2(f) = \sup_{w \in \boldsymbol{C}} f^{\#}(z)\,, \qquad \beta_2(r, f) = \sup_{w \in \boldsymbol{C}} T_1(r, f_{(w)})\,,$$

where

$$f_{(w)}(z) = f(z + w)\,, \qquad z \in \boldsymbol{C}\,.$$

THEOREM 2. *For f meromorphic in $\boldsymbol{C}$, and for r, $0 < r < \infty$, we have*

$$2r^{-2}\beta_2(r, f) \leqq \alpha_2(f)^2 \leqq r^{-2}[\exp\{2\beta_2(r, f)\} - 1]\,. \tag{1.2}$$

We call f a Yosida function, or a function of class (A) in K. Yosida's sense [7], if $\alpha_2(f) < \infty$. It then follows immediately from (1.2) that

$$\alpha_2(f) < \infty \,. \Leftrightarrow \beta_2(r, f) < \infty \qquad \textit{for each} \quad r \in (0, \infty)\,.$$
$$\Leftrightarrow \beta_2(r, f) < \infty \qquad \textit{for an} \quad r \in (0, \infty)\,.$$

If the right-hand term in (1.2) is bounded as $r \to \infty$, then f is rational (see [4, Theorem V.3, p. 197]), so that, $\alpha_2(f) < \infty$ trivially holds.

Thirdly, for f holomorphic in D, and for r, $0 < r \leqq 1$, we set

$$\alpha_3(f) = \sup_{w \in D} (1 - |w|^2)|f'(w)|\,, \qquad \beta_3(r, f) = \sup_{w \in D} T_2(r, f_w)\,.$$

THEOREM 3. *For f holomorphic in D, and for r, $0 < r < 1$, we have*

$$\left(\log \frac{1}{\sqrt{1 - r^2}}\right)^{-1} \beta_3(r, f) \leqq \alpha_3(f)^2 \leqq 2r^{-2}\beta_3(r, f)\,. \tag{1.3}$$

We call f Bloch if $\alpha_3(f) < \infty$ [3]. It then follows immediately from (1.3) that

$$\alpha_3(f) < \infty \,. \Leftrightarrow \beta_3(r, f) < \infty \qquad \textit{for each} \quad r \in (0, 1)\,.$$
$$\Leftrightarrow \beta_3(r, f) < \infty \qquad \textit{for an} \quad r \in (0, 1)\,.$$

If $\beta_3(1, f) < \infty$, then f is a BMOA function, and vice versa, (see, for example, [3]) because

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(\frac{re^{it} + w}{1 + \bar{w}re^{it}}\right) - f(w) \right|^2 dt = 2T_2(r, f_w)\,,$$

so that, it follows from the right-hand estimate of (1.3) that each BMOA function is Bloch as is observed in [3].

Finally, for f holomorphic and bounded, $|f| < 1$, in D, and for r, $0 < r \leqq 1$, we set

$$\alpha_4(f)=\sup_{w\in D}(1-|w|^2)f^*(w), \qquad \beta_4(r,f)=\sup_{w\in D} T_3(r,f_w).$$

In this case, it follows from the Schwarz-Pick lemma that $\alpha_4(f)\leqq 1$. Thus, $\beta_4(r,f)\leqq -(1/2)\log(1-r^2)$.

THEOREM 4. *For f holomorphic and bounded, $|f|<1$, in D, and for r, $0<r<1$, we have*

$$\left(\log\frac{1}{\sqrt{1-r^2}}\right)^{-1}\beta_4(r,f)\leqq\alpha_4(f)^2\leqq r^{-2}[1-\exp\{-2\beta_4(r,f)\}]. \tag{1.4}$$

We note that the right-most in (1.4) is not greater than one.

2. The mean-value expression

For u subharmonic in D_A, $0<A\leqq\infty$, we set its mean

$$M(r,u)=\frac{1}{2\pi}\int_0^{2\pi}u(re^{i\theta})d\theta, \qquad 0<r<A,$$

and $M(A,u)=\lim_{r\to A}M(r,u)\leqq\infty$.

In the case $A=1$ we set $u_w(z)=u((z+w)/(1+\bar{w}z))$, z, $w\in D$. Then $M(r,u_w)$ is the value at w of the least harmonic majorant of u in $\Delta(w,r)=\{z;\ |z-w|/|1-\bar{w}z|<r\}$, $0<r<1$. We suppose that u is so smooth that the Green formula holds:

$$r\frac{d}{dr}M(r,u_w)=\frac{1}{2\pi}\iint_{|z|<r}\Delta(u_w)(z)dxdy, \qquad 0<r<1.$$

On integrating we now obtain, for $0<r\leqq 1$,

$$M(r,u_w)-u(w)=\frac{1}{2\pi}\int_0^r t^{-1}\left[\iint_{|z|<t}\Delta(u_w)(z)dxdy\right]dt. \tag{2.1}$$

In the case $A=\infty$ we set $u_{(w)}(z)=u(z+w)$, z, $w\in \boldsymbol{C}$. Then under the obvious assumption on $u_{(w)}$ we have, for $0<r\leqq\infty$,

$$M(r,u_{(w)})-u(w)=\frac{1}{2\pi}\int_0^r t^{-1}\left[\iint_{|z|<t}\Delta(u_{(w)})(z)dxdy\right]dt. \tag{2.2}$$

The mean $M(r,u_{(w)})$ is the value at w of the least harmonic majorant of u in $D(w,r)=\{|z-w|<r\}$, $0<r<\infty$.

We begin with $T_1(r,f_w)$ and $T_1(r,f_{(w)})$. If f is meromorphic in D_A, then $f=f_1/f_2$, where f_1 and f_2 are holomorphic functions with no common zero in D_A. It is easy to see that $\Phi=\log(|f_1|^2+|f_2|^2)$ is subharmonic in D_A because $\Delta\Phi=4f^{\#2}$.

For $T_1(r,f_w)$ we can conclude from (2.1) that

$$2T_1(r,f_w)=M(r,\Phi_w)-\Phi(w), \qquad 0<r\leqq 1.$$

For $T_1(r, f_{(w)})$ we can conclude from (2.2) that

$$2T_1(r, f_{(w)}) = M(r, \Phi_{(w)}) - \Phi(w), \qquad 0 < r \leqq \infty .$$

To consider $T_2(r, f_w)$ we remark that $\Delta(|f|^2) = \Delta(|f-a|^2) = 4|f'|^2$ for all constants a, and we remember the consideration by Lehto [1]. On applying (2.1) to $u = |f|^2$ in D we obtain

$$\text{(2.3)} \qquad 2T_2(r, f_w) = M(r, |f_w|^2) - |f(w)|^2 = M(r, |f_w - f(w)|^2), \qquad 0 < r \leqq 1 .$$

Finally, to consider $T_3(r, f_w)$ we remark that $\Psi = -\log(1-|f|^2)$ is subharmonic in D because $\Delta\Psi = 4f^{*2}$. On applying (2.1) to $u = \Psi$ we obtain

$$\text{(2.4)} \qquad 2T_3(r, f_w) = M(r, \Psi_w) - \Psi(w), \qquad 0 < r \leqq 1 .$$

See [5] for related topics.

3. Proof of Theorem 1

To prove the left-hand side inequality of (1.1) we let $E \subset D$ and $0 < r < 1$. We shall prove

$$\text{(3.1)} \qquad \Delta(w, r) \subset E \Rightarrow T_1(r, f_w) \leqq \left(\log \frac{1}{\sqrt{1-r^2}}\right)\left[\sup_{z \in E}(1-|z|^2) f^{\#}(z)\right]^2 .$$

On setting $E = D$ we obtain the left half of (1.1). For the proof of (3.1) we suppose that the supremum a in the square brackets is finite. Then

$$\iint_{|z|<t} (f_w)^{\#}(z)^2 dxdy = \iint_{\Delta(w,t)} f^{\#}(z)^2 dxdy \leqq a^2 \iint_{\Delta(w,t)} (1-|z|^2)^{-2} dxdy = \frac{\pi a^2 t^2}{1-t^2},$$

whence

$$T_1(r, f_w) \leqq \frac{a^2}{2} \log \frac{1}{1-r^2} .$$

To prove the right-hand side inequality of (1.1) we let $0 < r < 1$. We then have

$$\text{(3.2)} \qquad zg(z) = \frac{f_w(rz) - f(w)}{1 + \overline{f(w)} f_w(rz)} = f_1(z)/f_2(z),$$

where g is meromorphic on $\bar{D} = \{|z| \leqq 1\}$, and f_1 and f_2 are holomorphic with no common zero on $\bar{D}$. The central term in (3.2) is interpreted as $1/f_w(rz)$ if $f(w) = \infty$. Then, $f_1(0) = 0 \neq f_2(0)$ and

$$\text{(3.3)} \qquad |g(0)| = r(1 - |w|^2) f^{\#}(w) .$$

Since there exists a holomorphic function h on $\bar{D}$ with $f_1(z) = zh(z)$, $z \in \bar{D}$, it follows that $g = h/f_2$. By the similar reasoning as in Section 2 we have

(3.4)
$$\frac{1}{2\pi}\int_0^{2\pi}\log(|f_1(e^{it})|^2+|f_2(e^{it})|^2)dt-2\log|f_2(0)|$$
$$=\frac{2}{\pi}\int_0^1\rho^{-1}\left[\iint_{|z|<\rho r}(f_w)^{\#}(z)^2dxdy\right]d\rho=2T_1(r,f_w).$$

On the other hand,

$$\log(1+|g(0)|^2)=\log(|h(0)|^2+|f_2(0)|^2)-2\log|f_2(0)|$$
$$\leqq\frac{1}{2\pi}\int_0^{2\pi}\log(|h(e^{it})|^2+|f_2(e^{it})|^2)dt-2\log|f_2(0)|.$$

Since $|h(e^{it})|=|f_1(e^{it})|$ it follows from (3.4) that

$$\log(1+|g(0)|^2)\leqq 2T_1(r,f_w),$$

which, together with (3.3) yields that

(3.5)
$$(1-|w|^2)^2f^{\#}(w)^2\leqq r^{-2}[\exp\{2T_1(r,f_w)\}-1].$$

The right-hand side of (1.1) now follows from (3.5).

Remark. With the aid of (3.1) and (3.5) we have, for f meromorphic in D, the following:

$$\lim_{|z|\to 1}(1-|z|^2)f^{\#}(z)=0.$$

$$\Leftrightarrow \lim_{|w|\to 1}T_1(r,f_w)=0 \quad \textit{for each} \quad r\in(0,1).$$

$$\Leftrightarrow \lim_{|w|\to 1}T_1(r,f_w)=0 \quad \textit{for an} \quad r\in(0,1).$$

We further remark that the equality in (3.5) for a pair w, r $(0<r<1)$, holds if and only if there exists a constant c such that

$$\frac{f(z)-f(w)}{1+\overline{f(w)}f(z)}=\frac{c}{r}\,\frac{z-w}{1-\bar{w}z}, \qquad z\in D.$$

Actually, the equality holds if and only if $\log(|h|^2+|f_2|^2)$ is harmonic or g is a constant c.

4. Proofs of Theorems 2, 3, and 4

The spirit of the proof is the same as in Section 3 in each case. It suffices to observe the analogies of type (3.1) and (3.5) which we denote by (3.1.k), (3.5.k), $k=2, 3, 4$.

Proof of Theorem 2. For $E\subset C$ and $0<r<\infty$, we have

$$(3.1.2) \qquad D(w, r) \subset E \Rightarrow T_1(r, f_{(w)}) \leqq 2^{-1} r^2 \left[\sup_{z \in E} f^\#(z) \right]^2 ,$$

and for $w \in \boldsymbol{C}$, $0 < r < \infty$, we have

$$(3.5.2) \qquad f^\#(w)^2 \leqq r^{-2}[\exp\{2T_1(r, f_{(w)})\} - 1] .$$

For the proof of (3.5.2) we consider

$$(3.2.2) \qquad zg(z) = \frac{f_{(w)}(rz) - f(w)}{1 + \overline{f(w)} f_{(w)}(rz)} = f_1(z)/f_2(z) \qquad \text{on} \quad \bar{D} .$$

Remark. For f meromorphic in $\boldsymbol{C}$, we have:

$$\lim_{|z| \to \infty} f^\#(z) = 0 .$$

$$\Leftrightarrow \lim_{|w| \to \infty} T_1(r, f_{(w)}) = 0 \qquad \textit{for each} \quad r \in (0, \infty) .$$

$$\Leftrightarrow \lim_{|w| \to \infty} T_1(r, f_{(w)}) = 0 \qquad \textit{for an} \quad r \in (0, \infty) .$$

We further remark that the equality in (3.5.2) for a pair w, r $(0 < r < \infty)$, holds if and only if there exists a constant c such that

$$\frac{f(z) - f(w)}{1 + \overline{f(w)} f(z)} = \frac{c}{r}(z - w) , \qquad z \in \boldsymbol{C} .$$

Proof of Theorem 3. For $E \subset D$ and $0 < r < 1$,

$$(3.1.3) \qquad \Delta(w, r) \subset E \Rightarrow T_2(r, f_w) \leqq \left(\log \frac{1}{\sqrt{1 - r^2}} \right) \left[\sup_{z \in E} (1 - |z|^2) |f'(z)| \right]^2 ,$$

and for $w \in D$, $0 < r < 1$,

$$(3.5.3) \qquad (1 - |w|^2)^2 |f'(w)|^2 \leqq 2r^{-2} T_2(r, f_w) .$$

For the proof of (3.5.3) we consider

$$(3.2.3) \qquad zg(z) = f_w(rz) - f(w) \qquad \text{on} \quad \bar{D} .$$

Then,

$$r^2(1 - |w|^2)^2 |f'(w)|^2 = |g(0)|^2 \leqq M(1, |g|^2) = M(r, |f_w - f(w)|^2) = 2T_2(r, f_w) .$$

Remark. For f holomorphic in D, we have:

$$\lim_{|z| \to 1} (1 - |z|^2) |f'(z)| = 0 .$$

$$\Leftrightarrow \lim_{|w| \to 1} T_2(r, f_w) = 0 \qquad \textit{for each} \quad r \in (0, 1) .$$

$$\Leftrightarrow \lim_{|w| \to 1} T_2(r, f_w) = 0 \qquad \textit{for an} \quad r \in (0, 1) .$$

We further remark that the equality in (3.5.3) for a pair w, r $(0<r<1)$, holds if and only if

$$f(z)-f(w)=\frac{c}{r}\frac{z-w}{1-\bar{w}z}, \qquad z\in D.$$

Proof of Theorem 4. For $E\subset D$ and $0<r<1$,

$$\text{(3.1.4)} \qquad \Delta(w,r)\subset E \Rightarrow T_3(r,f_w)\leqq\left(\log\frac{1}{\sqrt{1-r^2}}\right)\left[\sup_{z\in E}(1-|z|^2)f^*(z)\right]^2,$$

and for $w\in D$, $0<r<1$,

$$\text{(3.5.4)} \qquad (1-|w|^2)^2f^*(w)^2\leqq r^{-2}[1-\exp\{-2T_3(r,f_w)\}].$$

For the proof of (3.5.4) we consider

$$\text{(3.2.4)} \qquad zg(z)=\frac{f_w(rz)-f(w)}{1-\overline{f(w)}f_w(rz)} \qquad \text{on} \quad \bar{D}.$$

Then $|g(0)|=r(1-|w|^2)f^*(w)$, so that

$$\begin{aligned}-\log(1-|g(0)|^2)&\leqq M(1,\ -\log(1-|g|^2))\\ &=M\left(r,\ -\log\left(1-\left|\frac{f_w-f(w)}{1-\overline{f(w)}f_w}\right|^2\right)\right)=2T_3(r,f_w).\end{aligned}$$

Remark. For f holomorphic and bounded, $|f|<1$, in D we have:

$$\lim_{|z|\to 1}(1-|z|^2)f^*(z)=0.$$

$$\Leftrightarrow \lim_{|w|\to 1}T_3(r,f_w)=0 \qquad \textit{for each} \quad r\in(0,1).$$

$$\Leftrightarrow \lim_{|w|\to 1}T_3(r,f_w)=0 \qquad \textit{for an} \quad r\in(0,1).$$

We further remark that the equality in (3.5.4) for a pair w, r $(0<r<1)$, holds if and only if there exists a constant c, $|c|\leqq 1$, such that

$$\frac{f(z)-f(w)}{1-\overline{f(w)}f(z)}=\frac{c}{r}\frac{z-w}{1-\bar{w}z}, \qquad z\in D;$$

apparently, $|c|\leqq r$.

5. Further about Bloch functions

In view of the expression (2.3) and the criteria for $\alpha_3(f)<\infty$ just after Theorem 3, we easily find that the following gives more detailed criteria.

THEOREM 5. *For f holomorphic in D the following are equivalent.*

(5.1) *f is Bloch.*

(5.2) *For each* r, $0<r<1$, *we have*

$$\sup_{w\in D}\max_{|z|=r}|f_w(z)-f(w)|<\infty .$$

(5.3) *For an* r, $0<r<1$, *we have*

$$\sup_{w\in D} M(r, \log|f_w-f(w)|)<\infty .$$

Proof. (5.1) $\Rightarrow$ (5.2). It follows from

$$\sup_{z\in D}(1-|z|^2)|f'(z)|=\alpha<\infty$$

that

$$|f(z_1)-f(z_2)|\leqq\alpha\tanh^{-1}\left|\frac{z_1-z_2}{1-\bar{z}_1z_2}\right| \quad \text{for} \quad z_1, z_2\in D,$$

so that

$$|f_w(z)-f(w)|\leqq\alpha\tanh^{-1}r \quad \text{on} \quad |z|=r .$$

Since (5.2) $\Rightarrow$ (5.3) is trivial, it remains to be proved that (5.3) $\Rightarrow$ (5.1). For g of (3.2.3) we have

$$\log|g(0)|\leqq M(1, \log|g|)=M(r, \log|f_w-f(w)|) .$$

Consequently,

$$(1-|w|^2)|f'(w)|\leqq r^{-1}\exp M(r, \log|f_w-f(w)|) \quad \text{for all} \quad w\in D .$$

Remark. Similarly we have

$$\lim_{|z|\to 1}(1-|z|^2)|f'(z)|=0 .$$

$$\Leftrightarrow \lim_{|w|\to 1}\max_{|z|=r}|f_w(z)-f(w)|=0 \quad \textit{for each} \quad r\in(0, 1) .$$

$$\Leftrightarrow \lim_{|w|\to 1} M(r, \log|f_w-f(w)|)=-\infty \quad \textit{for an} \quad r\in(0, 1) .$$

Since $\alpha_4(f)\leqq 1$ for f holomorphic and bounded, $|f|<1$, in D, analogies of Theorem 5 in the present case are not interesting. However, the equivalence of the following is significant.

(I) $\lim_{|z|\to 1}(1-|z|^2)f^*(z)=0 .$

(II) $\lim_{|w|\to 1}\max_{|z|=r}\left|\frac{f_w(z)-f(w)}{1-\overline{f(w)}f_w(z)}\right|=0$ *for each* $r\in(0, 1)$.

(III) $\lim_{|w|\to 1} M\left(r, \log\left|\frac{f_w-f(w)}{1-\overline{f(w)}f_w}\right|\right)=-\infty$ *for an* $r\in(0, 1)$.

We can prove

$$2T_3(r,f_w)=M\left(r,\ \Psi\left(\left|\frac{f_w-f(w)}{1-\overline{f(w)}f_w}\right|\right)\right),$$

where $\Psi(x)=-\log(1-x^2)$, $0\leqq x<1$. Since $2\log x\leqq\Psi(x)$ and $\Psi(x)\to 0$ as $x\to 0$ we can easily observe that the criteria at the end of Section 4 follow. To prove (III) $\Rightarrow$ (I) we consider g of (3.2.4). Then,

$$\log|g(0)|\leqq M(1,\log|g|)=M\left(r,\log\left|\frac{f_w-f(w)}{1-\overline{f(w)}f_w}\right|\right).$$

The remaining details are left as exercises.

6. Comparisons of T_k, $k=1,2,3$

If f is holomorphic in D_A, $0<A\leqq\infty$, then

(6.1) $$T_1(r,f)\leqq T_2(r,f)\qquad(0<r\leqq A),$$

and if f is holomorphic and bounded, $|f|<1$, in D_A, then

(6.2) $$T_2(r,f)\leqq T_3(r,f)\qquad(0<r\leqq A).$$

We shall give more precise estimates than (6.1) and (6.2) in

THEOREM 6. *For f holomorphic in D_A we have*

(6.3) $$2T_1(r,f)\leqq\log\left[\frac{2T_2(r,f)}{1+|f(0)|^2}+1\right],\qquad 0<r\leqq A.$$

For f holomorphic and bounded, $|f|<1$, in D_A we have

(6.4) $$2T_2(r,f)\leqq(1-|f(0)|^2)[1-\exp\{-2T_3(r,f)\}],\qquad 0<r\leqq A.$$

Since for each constant a, $0\leqq a<\infty$, we have

$$\log\left\{\frac{2x}{1+a}+1\right\}\leqq 2x,\quad\text{and}\quad 1-e^{-2x}\leqq 2x\qquad\text{for}\quad 0\leqq x\leqq\infty,$$

we easily observe that (6.3) ((6.4), resp.) yields a better estimate than (6.1) ((6.2), resp.).

Proof of Theorem 6. To prove the first half we note that $\Delta\log(1+|f|^2)=4f^{\#2}$, whence

(6.5) $$2T_1(r,f)=M(r,\log(1+|f|^2))-\log(1+|f(0)|^2).$$

On the other hand, the geometric-arithmetic inequality yields

(6.6) $$\begin{aligned}M(r,\log(1+|f|^2))&\leqq\log M(r,1+|f|^2)\\&=\log\{M(r,|f|^2)+1\}=\log\{2T_2(r,f)+1+|f(0)|^2\}.\end{aligned}$$

We obtain (6.3) on combining (6.5) and (6.6).

To prove the second half we note that

$$2T_2(r, f) = M(r, |f|^2) - |f(0)|^2 . \tag{6.7}$$

By making use of Ψ in Section 5, we have

$$|f|^2 = 1 - \exp(-\Psi(|f|)) .$$

We then obtain

$$\begin{aligned} M(r, |f|^2) &\leqq 1 - \exp\{-M(r, \Psi(|f|)\} \\ &= 1 - \exp\{-2T_3(r, f) - \Psi(|f(0)|)\} , \end{aligned} \tag{6.8}$$

which, together with (6.7) and

$$1 - |f(0)|^2 = \exp\{-\Psi(|f(0)|)\} ,$$

yields (6.4).

Remark. The equality in (6.3) holds for an r, $0 < r < A$, if and only if f is a constant. Similarly, the equality in (6.4) holds for an r, $0 < r < A$, if and only if f is a constant.

As an application of (6.3) we remark that the $BMOA$ pseudo-norm $\|f\| \geqq 0$ of f holomorphic in D is given by

$$\|f\|^2 = 2 \sup_{w \in D} T_2(1, f_w) .$$

If $\|f\| < \infty$, then

$$2 \sup_{w \in D} T_1(1, f_w) \leqq \log\{\|f\|^2 + 1\} .$$

Therefore, if $f \in BMOA$, then $f \in UBC$, as is observed in [6] with a concrete norm relation. Similarly, we call f holomorphic in D to be of $VMOA$ if $\lim_{|w| \to 1} T_2(1, f_w) = 0$. With the aid of (6.1) we observe that f is a member of UBC_0 in the sense that

$$\lim_{|w| \to 1} T_1(1, f_w) = 0 .$$

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